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Modes of Convergence: Interpolation Methods I

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In the present paper we explore an approximation theoretic approach to some classical convergence theorems of real analysis. The background of this paper is the intuition that some of the usual compactness theorems on various modes of convergence in classical analysis are based on suitable ways of obtaining good decompositions of functions to exploit rates of approximation, cancellations, or appropriate control of sizes that can be controlled by the basic functionals of real interpolation. © 2001 Academic Press

1. INTRODUCTION

A perusal of some of the basic classical results relating norm convergence in L^1, convergence in measure, uniform integrability and weak compactness, suggests that a common method of analysis could be based on the functionals that govern the construction of real interpolation spaces. Indeed, real interpolation spaces are constructed using functionals that quantify precisely appropriate rates of approximation or best possible splittings of their elements.

In this paper we start the process of analyzing classical real variable convergence results using the methods of real interpolation. We hope to

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make the case that the methods of interpolation theory can be useful in this area and enlarge the scope and applications of the classical theory.[2]

In order to explain in more detail what we do let us recall the well known generalization of Lebesgue's dominated convergence theorem, due to Vitali, which states that given $\{f_n\}_{n\in N}\subset L^1$, $f\in L^1$, then[3] (cf. [9] page 180, and Lemma 2 below):

$$f_n \xrightarrow{L^1} f \Leftrightarrow \{f_n\}_{n\in N} \quad \text{is uniformly integrable and} \quad f_n \xrightarrow{m} f, \tag{1.1}$$

where $\xrightarrow{m}$ denotes convergence in measure. Comparing (1.1) with the classical Lebesgue dominated convergence theorem note that if there exists $g\in L^1$ such that $|f_n|\leqslant g$ for all n, then we obviously have that $\{f_n\}_{n\in N}$ is uniformly integrable. On the other hand it is well known, and easy to see, that uniform integrability *does not* imply pointwise domination. In a similar vein if we weaken pointwise domination to domination in the sense of distribution functions we still get a stronger condition than uniform integrability.[4] At this stage enter the K and E functionals of real interpolation theory for the pair (L^1, L^∞) (cf. Section 2 below). We have (cf. [2, 13], and the references therein)

$$K(t, f; L^1, L^\infty) = \int_0^t f^*(s)\, ds \tag{1.2}$$

$$\tilde{E}(t, f; L^1, L^\infty) = \int_t^\infty \lambda_f(s)\, ds, \tag{1.3}$$

where f^*, λ_f, denote respectively the decreasing rearrangement, and the distribution function of f. For $F\subset L^1$ we let

$$K(t, F; L^1, L^\infty) = \sup_{f\in F} K(t, f; L^1, L^\infty);$$

$$\tilde{E}(t, F; L^1, L^\infty) = \sup_{f\in F} \tilde{E}(t, f; L^1, L^\infty).$$

The rôle of the K and E functionals can be seen from the following statements (for proofs see Lemma 3 below)

[2] Some of our results can be in fact proved using the classical convergence theorems but we try to give an independent treatment.

[3] For the purpose of the exposition in this section the underlying measure space will be [0, 1] with Lebesgue measure.

[4] For example let $f_n = n\chi_{(0,\, 1/n\log n)}$, $n=3, \ldots$ then $\{f_n\}$ is uniformly integrable, but if g is such that $\lambda_{f_n}(t)\leqslant \lambda_g(t)$, for all $t>0$, $n=2, \ldots$ then $g\notin L^1$ (cf. [6]).

$$f_n \xrightarrow{m} f \Leftrightarrow \lim_{n\to\infty} -\tilde{E}'(t, f_n - f; L^1, L^\infty) = 0, \qquad t > 0 \tag{1.4}$$

(where $\tilde{E}' =$ a.e. the derivative of $\tilde{E}$[5]).

$$F \text{ is uniformly absolutely continuous} \Leftrightarrow \lim_{t\to 0} K(t, F; L^1, L^\infty) = 0. \tag{1.5}$$

$$F \text{ is uniformly integrable} \Leftrightarrow \lim_{t\to\infty} \tilde{E}(t, F; L^1, L^\infty) = 0. \tag{1.6}$$

But there is more. Let us say that $F \subset L^1$ is K-dominated if there exists $g \in L^1$ such that $K(t, F; L^1, L^\infty) \leqslant K(t, g; L^1, L^\infty)$. Now, if in the usual assumptions of Lebesgue's dominated convergence theorem we replace pointwise domination by K-domination we get the following form of the Lebesgue–Vitali theorem (cf. Theorem 8 below)

$$f_n \xrightarrow{L^1} f \Leftrightarrow \{f_n\}_{n\in N} \text{ is } K\text{-dominated and}$$

$$\lim_{n\to\infty} -\tilde{E}'(t, f_n - f; L^1, L^\infty) = 0, \qquad t > 0.$$

Let us show that in this classical context K-domination arises very naturally: indeed using (3.4) below we readily obtain for $0 < t \leqslant 1$,

$$K(t, \{f_n\}_{n\in N}, L^1, L^\infty) \leqslant \inf_{s>0} \{ts + \tilde{E}(s, \{f_n\}_{n\in N}; L^1, L^\infty)\} = \phi(t).$$

Now, if $\{f_n\}_{n\in N}$ is L^1 bounded then ϕ is quasi-concave and $\hat{\phi}$, the concave majorant of ϕ, is bounded on $[0, 1]$, moreover if $\lim_{s\to\infty} \tilde{E}(s, \{f_n\}_{n\in N}; L^1, L^\infty) = 0$ then we readily see that $\hat{\phi}(0^+) = 0$, thus we can write $\hat{\phi}(t) = \int_0^t \hat{\phi}'(s)\, ds$, with $\hat{\phi}'$ decreasing (cf. (3.11) below). Consequently $\hat{\phi}' \in L^1$ and

$$K(t, \{f_n\}_{n\in N}; L^1, L^\infty) \leqslant K(t, \hat{\phi}'; L^1, L^\infty).$$

In other words $\{f_n\}_{n\in N}$ is K-dominated by $\hat{\phi}'$. Note that the functionals associated to the real method of interpolation provide us with a constructive method, via Legendre transformation, to find a K-majorant for $\{f_n\}_{n\in N}$.

Once convergence problems have been formulated in this fashion the proofs depend on elementary properties of concave functions, Gagliardo diagrams, and their Legendre transformations. Moreover, *once formulated in the language of interpolation theory the Lebesgue–Vitali theorem can be*

[5] See the discussion in Section 3.

stated and proved in the general context of scales of real interpolation spaces.[6] As an application in Section 6 we derive versions of the Lebesgue–Vitali theorem in settings as diverse as the theory of Schatten ideals (non commutative integration), as well as the context of the variational problems studied by Michelli and Pinkus in [14].

The plan of the paper is as follows. In Section 2 we reformulate in detail the usual concepts of the classical theory (uniform integrability, uniform absolute integrability, convergence in measure, etc.) in terms of the K and E functionals. In Section 3 we review in detail the connection between K and E functionals. In Sections 4 and 5 we consider generalized versions of classical convergence theorems in the setting of scales of interpolation spaces. In Section 6 we consider applications including a version of the Lebesgue–Vitali convergence theorem for non commutative integration as well as version of the same theorem in the context of the variational problems studied by Michelli and Pinkus [14].

2. CLASSICAL THEORY

We start our presentation reformulating classical convergence theorems in the context of the Banach pair $(L^1, L^\infty) = (L^1(\Omega), L^\infty(\Omega))$, where (Ω, μ) is a finite measure space.[7] Recall that a subset $F \subset L^1 + L^\infty = L^1$, is said to be "*uniformly integrable*" iff $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\sup_{f \in F} \int_{\{|f| > \delta\}} |f(x)|\, d\mu(x) < \varepsilon.$$

In the literature one also finds the concept of "*uniform absolute continuity*" defined as follows: $F \subset L^1$ is uniformly absolutely continuous iff $\forall \varepsilon > 0$, $\exists \delta > 0$ such that for all measurable subsets $A \subset \Omega$ with $\mu(A) < \delta$ we have

$$\sup_{f \in F} \int_A |f(x)|\, d\mu(x) < \varepsilon.$$

[6] As we shall see, in the more general setting of interpolation spaces, K-domination is a delicate issue, which can be formulated as follows: which pairs of Banach spaces have the property that K-functionals of its elements generate sufficiently many concave functions? For example the argument given above shows that the pair (L^1, L^∞), has this property when the underlying measure space is $[0, 1]$, the case of general probability measure spaces is also true and can be easily reduced to the previous case via measure preserving transformations. More generally, the existence of sufficiently many concave functions associated with a given interpolation pair is connected with the deep part of real interpolation theory associated with "K-divisibility" (cf. [3] and the references therein, and Section 3 below).

[7] Unless otherwise specified all measure spaces in this paper are assumed to be finite measure spaces.

We also recall that a sequence of measurable functions $\{f_n\}_{n\in N}$ "*converges in measure*" to a measurable function f, briefly $f_n \xrightarrow{m} f$, iff $\forall t > 0$

$$\lim_{n\to\infty} \lambda_{f_n - f}(t) = 0,$$

where $\lambda_h(s) = \mu\{x: |h(x)| > s\}$ is the distribution function of h (by monotonicity it is easy to see that is enough for (2.1) to be valid a.e. $t > 0$).[8]

We now recall the classical results that motivated of our work[9] starting with the weak compactness theorem of Dunford Pettis [7]:

LEMMA 1. *Let (Ω, μ) be a finite measure space then*

$$F \subset L^1 \text{ is relatively weakly sequentially compact in } L^1$$

$$\Leftrightarrow F \text{ is uniformly integrable.}$$

As we remarked in the Introduction, uniform integrability also plays a rôle in the Lebesgue–Vitali convergence theorem.

LEMMA 2. *Let (Ω, μ) be a finite measure space, and let $\{f_n\}_{n\in N} \subset L^1$, $f \in L^1$, then*

$$f_n \xrightarrow{L^1} f \Leftrightarrow \{f_n\}_{n\in N} \quad \text{is uniformly integrable and} \quad f_n \xrightarrow{m} f.$$

A basic result due to Grothendieck (cf. [10]) connecting weak compactness with approximation properties was also a motivating factor in our research.

THEOREM 1 (cf. [10] p. 221). *Let H be a subset of a Banach space X such that for every $\varepsilon > 0$ there exists a weakly compact subset $H' \subset X$ such that for every $x \in H$, the distance of x to H' is smaller than ε. Then H is weakly relatively compact.*

In order to reformulate these concepts in terms of rates of approximation let us first review the definitions of some of the basic functionals of real interpolation. We consider compatible pairs of Banach spaces $\bar{A} = (A_0, A_1)$,

[8] Let us also recall that the decreasing rearrangement of h is given by $h^*(s) = \inf\{t: \lambda_h(t) \leqslant s\}$.

[9] Another early motivation to our work was Chaumat's extention of the Dunford–Pettis criterion. The second author is grateful to Aline Bonami and Jacques Chaumat for making [5] available to us and for several useful conversations. Weak compactness via interpolation methods will be studied in detail in the sequel to this paper.

that is we assume that there is a large topological vector space V such that $A_i \subset V$, $i=0, 1$, continuously. Usually we drop the terms "compatible" and "Banach" and refer to a compatible Banach pair simply as a "pair".

The K-functional associated with a pair $\bar{A}$ is defined, for $a \in A_0 + A_1 = \Sigma(\bar{A})$, $t > 0$, by

$$K(t, a) = K(t, a; A_0, A_1) = \inf\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} : a = a_0 + a_1, a_i \in A_i\}.$$

It is easy to see that $K(t, a)$ is a nonnegative, concave, increasing function of $t > 0$, (and thus also continuous). Furthermore $K(t, a+b) \leqslant K(t, a) + K(t, b)$, $a, b \in \Sigma(\bar{A})$, $t > 0$.

We shall say that the pair $\bar{A} = (A_0, A_1)$ is ordered[10] if $A_1 \subset A_0$, continuously and moreover $\|\cdot\|_{A_0} \leqslant \|\cdot\|_{A_1}$. In this case, for any $a \in A_0 + A_1 = A_0$, we have

$$K(t, a; A_0, A_1) = \|a\|_{A_0}, \qquad \forall t \geqslant 1.$$

The E-functional associated to a pair $\bar{A}$ is defined by

$$E(t, a) = E(t, a; A_0, A_1) = \inf\{\|a - a_0\|_{A_1} : \|a_0\|_{A_0} \leqslant t\}.$$

Observe that the E-functional is a nonnegative, convex, decreasing, continuous function of $t > 0$ which satisfies $E(2t, a+b) \leqslant E(t, a) + E(t, b)$, $a, b \in \Sigma(\bar{A})$, $t > 0$.

These definitions can be readily extended to subsets $F \subset A_0 + A_1$, thus we let

$$K(t, F) = K(t, F; A_0, A_1) = \sup_{a \in F} K(t, a)$$

$$E(t, F) = E(t, F; A_0, A_1) = \sup_{a \in F} E(t, a).$$

It will be also convenient to denote by $\tilde{K}$ (resp. $\tilde{E}$) the K-functional (resp. the E-functional) associated with the reverse pair (A_1, A_0), that is we let

$$\tilde{K}(t, a; A_0, A_1) = K(t, a; A_1, A_0) \quad \text{and} \quad \tilde{E}(t, a; A_0, A_1) = E(t, a; A_1, A_0).$$

For the pair (L^1, L^∞) the corresponding K and $\tilde{E}$-functionals are given respectively by (1.2) and (1.3). Using these explicit computations for the pair (L^1, L^∞) we can reinterpret uniform absolutely continuity, uniform integrability, and convergence in measure as follows

[10] For the most part in this paper we work with ordered pairs of Banach spaces.

LEMMA 3. *Let* $F \subset L^1 + L^\infty = L^1$ *and* $\{f_n\}_{n \in N} \subset L^1$, $f \in L^1$, *then,*

(1) *F is uniformly integrable* $\Leftrightarrow \lim_{t \to \infty} \tilde{E}(t, F) = 0$.

(2) $\sup_{f \in F} \|f\|_{L^1} \leqslant C$ *and F uniformly absolutely continuous* $\Leftrightarrow \lim_{t \to \infty} \tilde{E}(t, F) = 0$.

(3) $f_n \xrightarrow{m} f \Leftrightarrow \lim_{n \to \infty} -\tilde{E}'(t, f_n - f) = 0$, $t > 0$.

Proof. To prove 1 observe that

$$\begin{aligned}\int_{\{|f| > t\}} |f(x)| \, d\mu(x) &= \int_0^\infty \lambda_{(f\chi_{\{|f|>t\}})}(u) \, du \\ &= \int_t^\infty \lambda_f(u) \, du + t\lambda_f(t) \\ &\geqslant \tilde{E}(t, f) \qquad \text{(by (1.3)).}\end{aligned}$$

Thus, if F is uniformly integrable it follows that $\lim_{t \to \infty} \tilde{E}(t, F) = 0$. Conversely, suppose that $\lim_{t \to \infty} \tilde{E}(t, F) = 0$ then,

$$t\lambda_f(t) \leqslant 2 \int_{t/2}^t \lambda_f(u) \, du \leqslant 2\tilde{E}\left(\frac{t}{2}, f\right).$$

Consequently,

$$\int_{\{|f| > t\}} |f(x)| \, d\mu(x) = \int_t^\infty \lambda_f(u) \, du + t\lambda_f(t) \leqslant 3\tilde{E}\left(\frac{t}{2}, f\right),$$

and the uniform integrability of F follows.

2 follows readily from 1 and the fact (cf. [20] Theorem 2, p. 3) that F is uniformly integrable if and only if $\sup_{f \in F} \|f\|_{L^1} \leqslant C$ and F is uniformly absolutely continuous.

Finally by (1.3) we have

$$-\tilde{E}'(t, f_n - f) = \lambda_{f_n - f}(t) \qquad \text{a.e.} \quad t > 0,$$

and by monotonicity 3 follows. ∎

Remark 1. If (Ω, μ) is a *non-atomic* finite measure space, then we have (cf. [2, 4])

$$K(t, f; L^1, L^\infty) = \sup_{\mu(A) = t} \int_A |f(x)| \, d\mu = \sup_{\mu(A) \leqslant t} \int_A |f(x)| \, d\mu.$$

Therefore in this case if $F \subset L^1 + L^\infty = L^1$ we have,

$$F \text{ is uniformly absolutely continuous} \Leftrightarrow \lim_{t \to 0} K(t, F) = 0.$$

Remark 2. In Theorem 3 below we shall prove, in the general context of interpolation pairs,

$$\lim_{t \to \infty} \tilde{E}(t, F) = 0 \Leftrightarrow \lim_{t \to 0} K(t, F) = 0.$$

Thus for finite measure spaces, the following equivalences hold

(1) F is uniformly integrable,
(2) $\sup_{f \in F} \|f\|_{L^1} \leqslant C$ and F uniformly absolutely continuous,
(3) $\lim_{t \to \infty} \tilde{E}(t, F) = 0$,
(4) $\lim_{t \to 0} K(t, F) = 0$.

Summarizing our discussion we have

$$F \text{ is relatively weakly sequentially compact in } L^1 \Leftrightarrow \lim_{t \to 0} K(t, F) = 0$$

and

$$f_n \xrightarrow{L^1} f \Leftrightarrow \lim_{t \to 0} K(t, F) = 0 \qquad \text{and} \qquad \lim_{n \to \infty} -\tilde{E}'(t, f_n - f) = 0, \quad t > 0.$$

Before we give an extension of these results to the general context of interpolation theory we need to go somewhat deeper into the connections between the K and E functionals.

3. ON THE CONNECTION BETWEEN K AND E FUNCTIONALS

In this section we review some of the basic properties of the K and E functionals.[11] Our basic references here are [11], [2], [13], and [3].

We start recalling some elementary results from [11], [2] and [13]. Given a pair $\bar{A}$ we associate with $a \in \Sigma(\bar{A})$ a convex subset of R^2, $\Gamma(a)$ ($=$ the Gagliardo diagram of a) defined by

$$\Gamma(a) = \{(x_0, x_1) \in \mathbf{R}^2 : \exists a_i \in A_i \text{ s.t. } \|a_i\|_{Ai} \leqslant x_i,\ i = 0, 1;\ a = a_0 + a_1\}.$$

[11] The K-functional was apparently introduced independently by J. Peetre [18] and E. T. Oklander [17] around 1963, and was developed intensively afterwards by Peetre and his school. It is frequently refered to as Peetre's K-functional in order to reflect Peetre's fundamental and extensive contributions. Peetre also introduced the E and J functionals and the interpolation methods associated with these functionals.

Let $D(a)$ be defined by the intersection of the boundary of $\Gamma(a)$ and the nonnegative first quadrant:

$$D(a) = \partial\Gamma(a) \cap R^2_+ = \partial\Gamma(a) \cap \{(x_0, x_1) \in \mathbf{R}^2 : x_i > 0, i = 0, 1\}.$$

$D(a)$ may contain a semi-infinite vertical segment and/or semi-infinite horizontal segment. The remainder of the boundary will be the graph of a decreasing convex function (we suggest that the reader draws a picture).

The connection between the points $(x_0, x_1) \in D(a)$ and $K(t, a)$ is given by a kind of Legendre transform

$$K(t, a) = \inf_{(x_0, x_1) \in \Gamma(a)} \{x_0 + tx_1\} = \inf_{(x_0, x_1) \in D(a)} \{x_0 + tx_1\}, \tag{3.1}$$

i.e. $K(t, a)$ is the x_0-intercept of the tangent to $D(a)$ with slope $-1/t$. This follows from the fact that $K(t, a)$ is a nonnegative, increasing, concave and continuous function.

On the other hand, it follows readily from the definitions that the non-vertical part of the boundary of $D(a)$ is the curve

$$x_0 = r, \qquad x_1 = E(r, a).$$

Thus we can write (3.1) as

$$K(t, a) = \inf_{r > 0} \{r + tE(r, a)\}. \tag{3.2}$$

Now this means that at the points $r > 0$ where the derivative of $E(r, a)$ exists[12] we have

$$-\frac{1}{t} = E'(r, a), \qquad K(t, a) = r - \frac{E(r, a)}{E'(r, a)}. \tag{3.3}$$

Where the derivative does not exist then since $-1/t$ is between the left and the right derivatives of $E(r, a)$ we can give a meaning to (3.3) by letting $E'(r, a)$ take an appropriate value between the left and right derivatives.

Similarly, since the non-horizontal part of the boundary of $D(a)$ is the curve

$$x_0 = \tilde{E}(s, a), \qquad x_1 = s,$$

it follows from (3.1) that

$$K(t, a) = \inf_{s > 0} \{\tilde{E}(s, a) + ts\} \tag{3.4}$$

[12] Recall that since E is a convex function the derivative exists except perhaps for at most a countable set.

and therefore at the points $s>0$ where the derivative of $\tilde{E}(s,a)$ exists we have

$$t=-\tilde{E}'(s,a), \qquad K(t,a)=\tilde{E}(s,a)-\tilde{E}'(s,a)\,s. \tag{3.5}$$

As in the previous case we can give a meaning to (3.5) at those points s where $\tilde{E}'(s,a)$ does not exist by means of letting $\tilde{E}'(r,a)$ take an appropriate value between the right and left derivatives.

The inverse transform takes us back to the E-functional:

$$E(r,a)=\sup_{t>0}\left\{\frac{K(t,a)}{t}-\frac{r}{t}\right\}, \tag{3.6}$$

$$\tilde{E}(s,a)=\sup_{t>0}\{K(t,a)-ts\}. \tag{3.7}$$

Hence at the points $t>0$ where $K'(t,a)$ exists we find that

$$r=K(t,a)-K'(t,a)\,t, \qquad E(r,a)=K'(t,a); \tag{3.8}$$

$$s=K'(t,a), \qquad \tilde{E}(s,a)=K(t,a)-K'(t,a)\,t. \tag{3.9}$$

Using a by now familiar argument we can also give a meaning to $K'(t,a)$ even when $K(t,a)$ is not differentiable.

In particular, K' and $-\tilde{E}'$ are inverse to each other and $K-tK'$ and $-1/E'$ are inverse.

For example for the pair (L^1, L^∞) combining (1.2), (1.3) and (3.3) we obtain

$$\int_0^t f^*(s)\,ds-tf^*(t)=\int_{f^*(t)}^{\infty}\lambda_f(s)\,ds,$$

a well known and geometrically obvious formula relating $\int_0^t f^*, f^*, \lambda_f$.

In the sequel it will be also useful to have at hand some concepts from the calculus of convex functions (cf. [3]-Chapter 3 and the references quoted therein).

Let *Conv* denote the cone of all nonnegative concave functions on $R^+=(0,\infty)$, and let MC be the cone of all convex decreasing nonnegative functions on R^+.

Given a function $f\colon R^+\to R^+\cup\{0\}$ its *least concave majorant* is defined by

$$\hat{f}:=\inf\{g\in Conv : g\geqslant f\}. \tag{3.10}$$

A function $f\colon R^+ \to R^+ \cup \{0\}$ is *quasi-concave* if

$$f(t) \leqslant \max\left(1, \frac{t}{s}\right) f(s), \qquad s, t > 0.$$

If f is a quasi-concave function then f is equivalent to a concave function, more precisely we have

$$f \leqslant \hat{f} \leqslant 2f. \tag{3.11}$$

Similarly, given $f\colon R^+ \to R^+ \cup \{+\infty\}$ its *greatest convex minorant* is defined by

$$\check{f} := \inf\{g \in MC : g \leqslant f\}.$$

Obviously if $f(t) \neq \infty$ at least at a single point, then $\check{f} \neq \infty$, thus $\check{f} \in MC$.

Let $f\colon R^+ \to R^+ \cup \{+\infty\}$, by Legendre transformations we define

$$f^{\nabla}(t) := \inf_{s>0} \{f(s) + st\} \qquad \text{and} \qquad f^{\Delta}(t) := \sup_{s>0} \{f(s) - st\}.$$

It follows that

$$(f^{\Delta})^{\nabla} = \hat{f} \qquad \text{and} \qquad (f^{\nabla})^{\Delta} = \check{f}. \tag{3.12}$$

Let us also recall that $K(\cdot, a) \in Conv$, $E(\cdot, a) \in MC$ and

$$K(\cdot, a) = E(\cdot, a)^{\nabla}, \qquad E(\cdot, a) = K(\cdot, a)^{\Delta}.$$

We say that a pair $\bar{A}$ is *regular* if $\Delta(\bar{A}) = A_0 \cap A_1$ is dense in A_0. It follows (cf. [17]) that $\bar{A}$ is regular iff $\lim_{t \to 0} K(t, a) = 0$. In particular if $\bar{A}$ is regular, then

$$K(t, a) = \int_0^t K'(s, a)\, ds.$$

Note that if $\bar{A}$ is a regular ordered pair then, for all $a \in \Sigma(\bar{A}) = A_0$, we have

$$\int_0^1 K'(s, a)\, ds = \|a\|_{A_0} = \sup_{t>0} K(t, a). \tag{3.13}$$

DEFINITION 1. We say that $F \subset \Sigma(\bar{A})$ is K-bounded (resp E (resp $\tilde{E}$)-bounded) iff $\exists M > 0$, $t_0 > 0$ such that

$$K(t_0, F) \leqslant M$$

(resp $E(t_0, F) \leqslant M$ (resp $\tilde{E}(t_0, F) \leqslant M$)).

The next elementary result should be compared with Proposition (2.3.2) of [8].

Lemma 4. *F is K-bounded iff F is $\tilde{E}$-bounded iff F is E-bounded.*

Proof. Suppose that F is $\tilde{E}$-bounded, then there exist $M>0$, $t_0>0$, such that $\forall a \in F$,

$$\sup_t \{K(t, a) - tt_0\} = \tilde{E}(t_0, a) \leqslant M, \qquad \text{(by (3.7))}.$$

Therefore,

$$K(t_0, F) \leqslant M + t_0^2.$$

On the other hand, if F is K-bounded then, for some $M>0$, $t_0>0$, and all $a \in F$, we have

$$K(t_0, a) \leqslant M.$$

It follows that for each $a \in F$ we can find a decomposition $a = a_0 + a_1$ such that

$$\|a_0\|_{A_0} + t_0 \|a_1\|_{A_1} \leqslant 2M.$$

Consequently,

$$\|a - a_0\|_{A_1} \leqslant \frac{2M}{t_0}, \qquad \text{with} \quad \|a_0\|_{A_0} \leqslant 2M,$$

and

$$\tilde{E}(2M, F) \leqslant \frac{2M}{t_0}.$$

Finally observe that since we trivially have that F is K-bounded if and only if F is $\tilde{K}$-bounded all the equivalences of the Lemma have been proved. ∎

Remark 3. Note that if $\bar{A}$ is an ordered pair then for each fixed $t>0$, $K(t, .)$ defines an equivalent norm on A_0. Thus, $F \subset A_0$ is K-bounded iff

$$K(t, F) = \sup_{a \in F} \|a\|_{A_0} < \infty, \qquad t \geqslant 1.$$

4. UNIFORM CONTINUITY OF K AND E FUNCTIONALS

Definition 2. Let $\bar{A} = (A_0, A_1)$ be a pair, we shall say that $F \subset A_0 + A_1$ is K-uniformly continuous (at 0) iff

$$\lim_{t \to 0} K(t, F) = 0.$$

We say that F is E-uniformly continuous (at ∞) iff

$$\lim_{t\to\infty} E(t, F)=0.$$

We say that F is K-dominated iff there exists $a\in A_0+A_1$ such that

$$K(t, F)\leqslant K(t, a), \qquad \forall t>0.$$

Similar definitions for uniform continuity at other points or other functionals can be given.

Remark 4. Obviously K-uniform continuity implies K-boundedness and therefore by Remark 3 it follows that for ordered pairs K-uniform continuity implies $K(t, F)<\infty$, $t>0$. Moreover note that from $K(t, a)\leqslant \max(1, \frac{t}{s})\times K(s, a)$ it follows that $K(t, F)\leqslant \max(1, \frac{t}{s})\, K(s, F)$, i.e. $K(t, F)$ is quasi-concave. If the pair is regular then K-domination implies K-uniform continuity.

The following results follow directly from the definitions.

PROPOSITION 1. *Let $\bar{A}$ be an ordered pair, and suppose moreover that $i\colon A_1\subset A_0$ is weakly compact. Suppose that $F\subset A_0$ is $\tilde{E}$-uniformly continuous then F is weakly relatively compact.*

Proof. Let $\varepsilon>0$ be given and choose $\delta>0$ so large that

$$\tilde{E}(\delta, F)<\varepsilon.$$

Therefore each $a\in F$ is at distance less than ε of the A_1 ball $B(0, \delta)$. We may now apply Grothendieck's lemma to conclude (cf. Theorem 1). ▮

PROPOSITION 2. *Let $\bar{A}$ and $\bar{B}$ be pairs and let $T\colon \bar{A}\to\bar{B}$ be an operator (possibly non-linear) such that there exists $c>0$ such that $K(t, Ta; \bar{B})\leqslant cK(t, a; \bar{A})$, $\forall a\in\Sigma(\bar{A})$ (i.e. T is a K-bounded (resp E-bounded) operator). Then if $F\subset\Sigma(\bar{A})$ is K-uniformly continuous (resp E-uniformly continuous) it follows that $T(F)\subset\Sigma(\bar{B})$ is K-uniformly continuous (resp E-uniformly continuous).*

Proof. The result follows immediately from

$$K(t, Ta; \bar{B})\leqslant cK(t, a; \bar{A})\leqslant cK(t, F; \bar{A})$$

$$\sup_{b\in T(F)} K(t, b; \bar{B})\leqslant cK(t, F; \bar{A}).$$

A similar remark proves the assertion on E-uniform continuity.

COROLLARY 1. *If $T: \bar{A} \to \bar{B}$ is a bounded linear operator, then T is K-bounded (resp E-bounded) and the previous result applies.*

The theory of commutators for the real method of interpolation (cf. [15] and [16] for recent accounts) provides us with a set of examples. We state here a result in terms of E functionals, similar results hold for the K method.

PROPOSITION 3. *Let $\bar{A}$ and $\bar{B}$ be Banach pairs and let T be a bounded operator, $T: \bar{A} \to \bar{B}$, then if $F \subset \Sigma(\bar{A})$ is such that $\lim_{t\to\infty} \int_t^\infty E(s, F; \bar{A}) \frac{ds}{s} = 0$, then $[T, \Omega_E](F)$ is E-uniformly continuous.*

Proof. It is known (cf. [15]) that there exist constants $c, c' > 0$, such that $\forall a \in F$ we have,

$$E(2ct, [T, \Omega]\, a; \bar{B}) \leqslant c' \int_t^\infty E(s, a; \bar{A}) \frac{ds}{s} \leqslant c' \int_t^\infty E(s, F; \bar{A}) \frac{ds}{s}.$$

Consequently,

$$E(2ct, [T, \Omega]\, F; \bar{B}) \leqslant c' \int_t^\infty E(s, F; \bar{A}) \frac{ds}{s},$$

and the result follows. ▮

In order to establish the connection between K-uniform continuity, $\tilde{E}$-uniform continuity and K-domination we need a result from [3]. First some notation from [3].

DEFINITION 3. Let $I_- = (0, 1]$. We say that a pair $\bar{A}$ is *Conv*$_-$*-abundant* if for all $\varphi \in Conv$ such that $\lim_{t\to 0} \varphi(t) = 0$ there exists $x \in \Sigma(\bar{A})$ such that

$$K(t, x) \approx \varphi(t), \qquad t \in I_-.$$

The next result from [3] gives a criterion to determine when a pair is *Conv*$_-$*-abundant*.

THEOREM 2 (cf. [3] Theorem 4.5.7). *Let $\bar{A} = (A_0, A_1)$ be a pair. Assume that there exists a nonzero $a_0 \in \Sigma(\bar{A})$ such that for all $t \in I_-$*

$$K(t, a_0) \approx \int_0^t K(s, a_0) \frac{ds}{s} + t \int_t^\infty K(s, a_0) \frac{ds}{s^2}.$$

Then $\bar{A}$ is Conv$_-$-abundant.

We now show that, under a *Conv*$_-$ abundance assumption, K-uniform continuity is equivalent to K-domination.

THEOREM 3. *Let $\bar{A} = (A_0, A_1)$ be a ordered pair, and let $F \subset \Sigma(\bar{A})$ then*

$$F \text{ is } K\text{-uniformly continuous} \Leftrightarrow F \text{ is } \tilde{E}\text{-uniformly continuous.}$$

Moreover if $\bar{A}$ is regular and Conv$_-$-abundant *then*

$$F \text{ is } K\text{-uniformly continuous} \Leftrightarrow F \text{ is } K\text{-dominated.}$$

For the proof of Theorem 3 we need the following

LEMMA 5. *Let h be a nonnegative and decreasing function then*

$$\lim_{t \to \infty} h(t) = \lim_{t \to 0} \inf_{s > 0} (h(s) + st).$$

Proof. Since h is decreasing $\lim_{t \to \infty} h(t) = \inf_{t > 0} h(t)$. For all $s > 0$, $t > 0$,

$$\inf_{t > 0} h(t) \leqslant h(s) + st.$$

Therefore,

$$\inf_{t > 0} h(t) \leqslant \lim_{t \to 0} \inf_{s > 0} (h(s) + st).$$

Conversely, for any $s, t > 0$,

$$\inf_{s > 0} (h(s) + st) \leqslant h(s) + st,$$

implies

$$\lim_{t \to 0} \inf_{s > 0} (h(s) + st) \leqslant h(s)$$

$$\lim_{t \to 0} \inf_{s > 0} (h(s) + st) \leqslant \inf_{s > 0} h(s). \quad \blacksquare$$

We are now ready for the proof of Theorem 3:

Proof. First assume that F is K-uniformly continuous, then by Remark 4, $K(z, F) < \infty$, and by (3.7) we have

$$\tilde{E}(s, F) \leqslant \sup_{z > 0} \{ \sup_{a \in F} K(z, a) - sz \} = \sup_{z > 0} \{ K(z, F) - zs \} < \infty. \tag{4.1}$$

Furthermore, since $\widetilde{E}(t, a)$ is nonnegative and decreasing it follows that $\widetilde{E}(t, F)$ is also nonnegative and decreasing, thus by Lemma 5

$$\begin{aligned}\lim_{t\to\infty} \widetilde{E}(t, F) &= \lim_{t\to 0} \inf_{s>0} (\widetilde{E}(s, F) + st)\\ &\leqslant \lim_{t\to 0} \inf_{s>0} \{\sup_{z>0} \{K(z, F) - zs\} + st\} \qquad \text{(by (4.1))}\\ &= \lim_{t\to 0} [K(\cdot, F)^{\Delta}]^{\nabla}(t) = \lim_{t\to 0} \widehat{K(t, F)} \qquad \text{(by (3.12))}.\end{aligned}$$

By Remark 4, $K(t, F)$ is quasi-concave, therefore by (3.11)

$$K(t, F) \leqslant \widehat{K(t, F)} \leqslant 2K(t, F).$$

Since F is K-uniformly continuous we see that

$$\lim_{t\to\infty} \widetilde{E}(t, F) \leqslant \lim_{t\to 0} \widehat{K(t, F)} \leqslant 2 \lim_{t\to 0} K(t, F) = 0.$$

Conversely by (3.4)

$$\begin{aligned}K(t, a) &= \inf_{s>0} \{\widetilde{E}(s, a) + ts\} \leqslant \inf_{s>0} \{\sup_{a\in F} \widetilde{E}(s, a) + ts\}\\ &= \inf_{s>0} \{\widetilde{E}(s, F) + ts\},\end{aligned}$$

hence

$$K(t, F) \leqslant \inf_{s>0} \{\widetilde{E}(s, F) + ts\} \leqslant \widetilde{E}(s, F) + ts.$$

It follows that

$$\lim_{t\to 0} K(t, F) \leqslant \lim_{t\to 0} (\widetilde{E}(s, F) + ts) = \widetilde{E}(s, F).$$

Now, since F is $\widetilde{E}$-uniformly continuous

$$\lim_{t\to 0} K(t, F) \leqslant \lim_{s\to\infty} \widetilde{E}(s, F) = 0.$$

Suppose now that $\bar{A}$ is regular and *Conv*$_{-}$*-abundant*. Since F is K-uniformly continuous, $K(t, F)$ is quasi-concave (cf. Remark 4) therefore $\widehat{K(t, F)}$ is well defined. Furthermore, (cf. (3.11))

$$\lim_{t\to 0} \widehat{K(t, F)} = 0.$$

Theorem 4.2 implies the existence of $f \in \Sigma(\bar{A})$ such that

$$\widehat{K(t, F)} \approx K(t, f), \qquad 0 \leqslant t \leqslant 1,$$

hence

$$K(t, F) \leqslant \widehat{K(t, F)} \approx K(t, f), \qquad 0 \leqslant t \leqslant 1,$$

as we wished to show. ▮

We close this section with an abstract version of the La Vallée Poussin Criteria for uniform integrability.

THEOREM 4. *Let $\bar{A}$ be an ordered regular pair, and let $F \subset A_0$. Then, the following are equivalent*

(i) *F is $\tilde{E}$-uniformly continuous*

(ii) $\exists \phi$: $(0, \infty) \to (0, \infty)$, *with* $\lim_{t \to \infty} \phi(t) = \infty$, *such that*

$$\sup_{f \in F} \int_0^\infty (-\tilde{E}'(f, s))\, \phi(s)\, ds = M < \infty.$$

Proof. Suppose that (ii) holds. By our assumption on the pair $\bar{A}$, for every $a \in A_0$ we have

$$\lim_{t \to \infty} \tilde{E}(t, a) = 0.$$

Thus, for all $a \in F$, $\forall \delta > 0$, we have

$$\tilde{E}(\delta, a) = \int_\delta^\infty (-\tilde{E}'(a, s))\, ds.$$

Let $\varepsilon > 0$, and choose $\delta > 0$ such that $\phi(u) > \frac{M}{\varepsilon}$, whenever $u > \delta$. Then,

$$\tilde{E}(\delta, a) \leqslant \frac{\varepsilon}{M} \int_\delta^\infty (-\tilde{E}'(f, a))\, \phi(s)\, ds \leqslant \varepsilon.$$

Taking supremum over all $a \in F$,

$$\tilde{E}(\delta, F) \leqslant \varepsilon,$$

as we wished to show.

Conversely, suppose that F is $\tilde{E}$-uniformly continuous, we now argue as in the usual proof of the De La Vallée Poussin criteria (cf. [8]) to construct ϕ. In fact, since $\tilde{E}(\cdot, F)$ is decreasing and $\lim_{\delta \to \infty} \tilde{E}(\delta, F) = 0$, we can choose δ_0 such that $\widetilde{E}(\delta_0, F) < 1$, then $-\log \widetilde{E}(\delta, F)$ is a nonnegative increasing function for $\delta \geqslant \delta_0$ and $\lim_{\delta \to \infty} -\log \widetilde{E}(\delta, F) = \infty$. Let ϕ: $(0, \infty) \to (0, \infty)$ be any continuous function, such that $\phi(\delta) = 0$, $\forall \delta \in (0, \delta_0)$, and $\phi(\delta) \leqslant -\log \widetilde{E}(\delta, F)$, if $\delta \geqslant \delta_0$, ϕ strictly increasing on (δ_0, ∞), and $\lim_{\delta \to \infty} \phi(\delta) = \infty$. As a consequence of this construction we see that $\widetilde{E}(\phi^{-1}(u), F) \leqslant e^{-u}$, for all $u > 0$. Now, let $a \in F$, then

$$\begin{aligned}
\int_0^\infty (-\widetilde{E}'(\delta, a))\, \phi(\delta)\, d\delta &= \int_0^\infty (-\widetilde{E}'(\delta, a)) \int_0^{\phi(\delta)} du\, d\delta \\
&= \int_0^\infty \int_{\phi^{-1}(u)}^\infty (-\widetilde{E}'(\delta, a))\, d\delta\, du \\
&= \int_0^\infty \widetilde{E}(\phi^{-1}(u), a)\, du \\
&\leqslant \int_0^\infty \widetilde{E}(\phi^{-1}(u), F)\, du \\
&= \int_0^\infty e^{-u}\, du = 1.
\end{aligned}$$

Taking supremum over all $a \in F$ we see that (ii) holds and the desired result follows. ▮

Example 1. For comparison let us recall the classical De La Vallée Poussin criteria. Let (Ω, μ) be a finite measure space and consider the pair (L^1, L^∞). $F \subset L^1$ is uniformly integrable iff there exists a finite Orlicz function A such that $\lim_{u \to \infty} \frac{A(u)}{u} = \infty$ and F is a bounded set in the Orlicz space L_A. In order to recover this result from Theorem 4 we just need to remark that for an Orlicz function A, we have, by Fubini's theorem,

$$\int_\Omega A(|f(x)|)\, d\mu(x) = \int_0^\infty \lambda_f(t)\, A'(t)\, dt$$

and that $\lim_{t \to \infty} A'(t) = \infty$ whenever $\lim_{t \to \infty} \frac{A(t)}{t} = \infty$.

5. CONVERGENCE PROCESSES ASSOCIATED WITH DERIVATIVES OF FUNCTIONALS

The model result we wish to extend in this section is the Lebesgue–Vitali Lemma (cf. Lemma 2).

THEOREM 5. *Let* (A_0, A_1) *be an ordered regular pair, let* $\{a_n\}_{n\in N} \subset A_0$ *and* $a \in A_0$, *then*

$$a_n \xrightarrow{A_0} a \Leftrightarrow \{a_n\}_{n\in N} \text{ is } K\text{-uniformly continuous and}$$
$$\lim_{n\to\infty} -\tilde{E}'(s, a-a_n) = 0, \qquad s>0.$$

Proof. Let us start by remarking that in view of (3.5) and (3.9), for any $a \in A_0$,

$$\begin{aligned}
\sup_{t>0} t(-\tilde{E}'(t,a)) &= \sup_{t>0} tK'(t,a) \\
&\leqslant \sup_t \int_0^t K'(s,a)\,ds \qquad &&\text{(since } K' \text{ decreases)} \\
&= \sup_t K(t,a) &&\text{(since the pair is regular)} \\
&= \|a\|_{A_0} &&\text{(by (3.13)).} \qquad (5.1)
\end{aligned}$$

Suppose now that $a_n \xrightarrow{A_0} a$. Then, by (5.1), $\forall t>0$,

$$-\tilde{E}'(t, a_n - a) \leqslant \frac{1}{t}\,\|a_n - a\|_{A_0}$$

and therefore

$$\lim_{n\to\infty} -\tilde{E}'(t, a_n - a) = 0.$$

Let us now prove that $\{a_n\}_{n\in N}$ is K-uniformly continuous. Let $\varepsilon>0$, and choose n_0 large enough so that $\|a_n - a\|_{A_0} < \frac{\varepsilon}{2}$, for $n>n_0$, then $\forall t>0$, $n>n_0$, we have

$$K(t, a_n - a) \leqslant \|a_n - a\|_{A_0} < \frac{\varepsilon}{2}. \qquad (5.2)$$

Now select $t_0(n_0)>0$ sufficiently small so that if $t<t_0$

$$\max_{n=1\cdots n_0} K(t, a_n - a) + K(t,a) < \frac{\varepsilon}{2}. \qquad (5.3)$$

(Note that, since the pair is regular, sets with a finite number of elements are K-uniformly continuous). Then, combining (5.2) and (5.3) with the triangle inequality, we have, for $n \in N$, $t<t_0$,

$$K(t, a_n) \leqslant K(t, a_n - a) + K(t,a) < \varepsilon,$$

proving the K-uniform continuity.

Suppose now that $\lim_{n\to\infty} -\tilde{E}'(s, a-a_n)=0$, $\forall s>0$ and $\{a_n\}_{n\in N}$ is K-uniformly continuous. Let $\varepsilon>0$ be given and select $t_0<1$ so that on account of the K-uniform continuity we have $\forall n\in N$,

$$K(t_0, a_n-a)<\frac{\varepsilon}{2}.$$

Select n_0 such that $\forall n>n_0$,

$$-\tilde{E}'\left(\frac{\varepsilon}{2}, a-a_n\right)<t_0.$$

Now, let us write

$$\begin{aligned}\|a-a_n\|_{A_0} &= \int_0^1 K'(s, a-a_n)\,ds \\ &= \int_{\{s\in(0,1):K'(s,a-a_n)>\varepsilon/2\}} K'(s, a-a_n)\,ds \\ &\quad + \int_{\{s\in(0,1):K'(s,a-a_n)\leqslant\varepsilon/2\}} K'(s, a-a_n)\,ds \\ &= I+II.\end{aligned}$$

It is plain that

$$II\leqslant\varepsilon/2.$$

To estimate I, let us recall again that the inverse of $K'(s, a-a_n)$ is the decreasing function $-\tilde{E}'(s, a-a_n)$, thus we see that

$$\{s: K'(s, a-a_n)>\varepsilon/2\}=\{s: s<-\tilde{E}'(\varepsilon/2, a-a_n)\}.$$

Therefore,

$$\begin{aligned}I &= \int_0^{-\tilde{E}'(\varepsilon/2, a-a_n)} K'(s, a-a_n)\,ds \\ &= K(-\tilde{E}'(\varepsilon/2, a-a_n), a-a_n) \\ &\leqslant K(t_0, a-a_n) \qquad \text{(if } n>n_0\text{, since } K \text{ increases)} \\ &\leqslant \varepsilon/2.\end{aligned}$$

Combining estimates we get that for $n > n_0$ it holds

$$\|a - a_n\|_{A_0} \leqslant \varepsilon,$$

as we wished to show. ∎

Remark 5. In general the assumptions that the pair (A_0, A_1) be ordered and regular cannot be dispensed with. For example, consider the ordered non regular pair (ℓ^∞, ℓ^1) and let $\{a_n\}_{n \in N} \in \ell^\infty$ be defined by

$$a_n^m = \frac{1}{n}, \qquad m = 1, 2...$$

We obviously have

$$a_n \xrightarrow{\ell^\infty} 0.$$

However since,

$$K(t, a_n, \ell^\infty, \ell^1) = tK\left(\frac{1}{t}, a_n, \ell^1, \ell^\infty\right) = t \sum_{m=1}^{[1/t]} \frac{1}{n}$$

($[t] :=$ integer part of t) it follows that

$$\lim_{t \to 0} \sup_n K(t, a_n, \ell^\infty, \ell^1) = \lim_{t \to 0} \sup_n \left(t \sum_{m=1}^{[1/t]} \frac{1}{n}\right) = 1.$$

Consider the non-ordered pair $(L^1[0, \infty], L^\infty[0, \infty])$ it is easy to construct a sequence $\{f_n\}_{n \in N} \subset L^1[0, \infty]$, $f \in L^1[0, \infty]$ such that

$$\{f_n\}_{n \in N} \qquad \text{is } K\text{-dominated and} \quad f_n \xrightarrow{m} f$$

but

$$f_n \overset{L^1}{\not\longrightarrow} f.$$

Effectively, consider $f_n(t) = \frac{1}{n} \chi_{[0, n]}(t)$, then $f_n \xrightarrow{m} 0$. Moreover

$$K(t, f_n) = \begin{cases} \dfrac{t}{n}, & \text{for} \quad t < n \\ 1, & \text{for} \quad t \geqslant n \end{cases}$$

and therefore

$$K(t, f_n) \leqslant K(t, f_1), \qquad n = 1...$$

but

$$\|f_n\|_{L^1} = 1, \qquad n = 1...$$

Remark 6. As the referee has kindly pointed out it is easy to reformulate our convergence results in terms of E convergence and domination. We leave the formulation of such results to the interested reader.

5.1. *Reiteration.* The classical Lebesgue–Vitali theorem for L^p spaces involves convergence in measure (i.e. $f_n \xrightarrow{m} f$, or $-\tilde{E}'(t, f_n - f; L^1, L^\infty) \to 0$) while our formulation requires $-\tilde{E}'(t, f_n - f; L^p, L^\infty) \to 0$. In this section we discuss briefly the rôle of reiteration in the study of convergence in interpolation scales. We will formulate the results in terms of the Lions–Peetre scale of real interpolation spaces (cf. [2]). Recall that given a pair $\bar{A}$, and $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, we let

$$\bar{A}_{\theta, q; K} = \left\{ a \in \Sigma(\bar{A}) : \|a\|_{\bar{A}_{\theta, q; K}} = \left\{ \int_0^\infty (s^{-\theta} K(s, a; \bar{A}))^q \frac{ds}{s} \right\}^{1/q} < \infty \right\}.$$

If $\bar{A}$ is an ordered pair $\bar{A}_{\theta, q; K}$ can be equivalently renormed by

$$\|a\|_{\bar{A}_{\theta, q; K}} = \left\{ \int_0^1 (s^{-\theta} K(s, a; \bar{A}))^q \frac{ds}{s} \right\}^{1/q}.$$

THEOREM 6. *Let $\bar{A} = (A_0, A_1)$ be an ordered regular pair, $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, and let $\{a_n\}_{n \in N} \subset \bar{A}_{\theta, q; K}$, $a \in \bar{A}_{\theta, q; K}$. Suppose that $\lim_{n \to \infty} -\tilde{E}'(s, a - a_n; \bar{A}) = 0$, a.e. $s > 0$, and that $\{a_n\}_{n \in N}$ is $K - (\bar{A}_{\theta, q; K}, A_1)$ uniformly continuous, then*

$$a_n \xrightarrow{A_{\theta, q; K}} a.$$

Proof. Recall that by Holmstedt's reiteration formula (cf. [2] Corollary 3.6.2) we have,

$$K(t, a; \bar{A}_{\theta, q; K}, A_1) \approx \left\{ \int_0^{t^{1/(1-\theta)}} [s^{-\theta} K(s, a; \bar{A})]^q \frac{ds}{s} \right\}^{1/q}. \tag{5.4}$$

Let $\varepsilon > 0$, and let $t_0 \in (0, 1)$ to be chosen precisely later. Then for all $n \in N$,

$$\begin{aligned} \|a_n - a\|_{\bar{A}_{\theta, q; K}} &= \left\{ \int_0^1 [s^{-\theta} K(s, a_n - a; \bar{A})]^q \frac{ds}{s} \right\}^{1/q} \\ &\leqslant c \sup_n K(t_0, a_n - a; \bar{A}_{\theta, q; K}, A_1) \\ &\quad + c \left\{ \int_{t_0^{1/(1-\theta)}}^1 [s^{-\theta} K(s, a_n - a; \bar{A})]^q \frac{ds}{s} \right\}^{1/q} \qquad \text{(by (5.4))} \\ &= I + II. \end{aligned}$$

Using the fact that $\{a_n\}_{n \in N}$ is $K-(\bar{A}_{\theta, q; K}, A_1)$ uniformly continuous we can now choose t_0 so that

$$I < \varepsilon/2.$$

On the other hand since $K(s, .)/s$ is decreasing we have

$$II \leqslant c_{\theta, q} K(t_0^{1/(1-\theta)}, a_n - a; \bar{A})\, t_0^{-1/(1-\theta)}.$$

It follows from (5.4) that $K-(\bar{A}_{\theta, q; K}, A_1)$ uniform continuity implies $K-(A_0, A_1)$ uniform continuity, moreover since by hypothesis we have $\lim_{n \to \infty} -\tilde{E}'(s, a - a_n; \bar{A}) = 0$, $s > 0$, we conclude from Theorem 5 that

$$\|a_n - a\|_{A_0} \to 0.$$

Observe that,

$$K(t_0^{1/(1-\theta)}, a_n - a; \bar{A}) \leqslant \|a_n - a\|_{A_0}.$$

Therefore it follows that we can select $n_0 \in N$ such that for all $n > n_0$ we have

$$K(t_0^{1/(1-\theta)}, a_n - a; \bar{A}) \leqslant \frac{\varepsilon t_0^{1/(1-\theta)}}{2c_{\theta, q}}.$$

Combining estimates we see that for all $n > n_0$

$$\|a_n - a\|_{\bar{A}_{\theta, q; K}} < \varepsilon,$$

as desired. ▮

K-domination can be also sharpened by reiteration.

PROPOSITION 4. *Let $\bar{A}$ be an ordered pair and let $F \subset A_0$ be K-dominated by $g \in \bar{A}_{\theta, q; K}$. Then, $F \subset \bar{A}_{\theta, q; K}$, and F is $K-(\bar{A}_{\theta, q; K}, A_1)$ dominated by g.*

Proof. Direct consequence of Holmstedt's reiteration formula. ▮

EXAMPLE 2. Let (Ω, μ) be a finite measure space. Suppose that $\{f_n\}_{n \in N} \subset L^p$, $f \in L^p$. If $f_n \xrightarrow{m} f$, and $\{|f_n|^p\}_{n \in N}$ is uniformly integrable then $f_n \xrightarrow{L^p} f$.

Proof. Recall that (cf. [2])

$$K(t, f; L^p, L^\infty) \approx \left\{\int_0^{t^p} f^{*p}(s)\, ds\right\}^{1/p}.$$

Therefore $\{|f_n|^p\}_{n \in N}$ is uniformly integrable iff $\{f_n\}_{n \in N}$ is $K-(L^p, L^\infty)$ uniformly continuous. We conclude applying Theorem 6. ▮

5.2. *General pairs.* We consider the modifications that are necessary in order to deal with pairs that are not ordered. We shall consider pairs that are *mutually closed*, that is $\forall a \in \Sigma(\bar{A})$

$$\lim_{t\to\infty} K(t, a; \bar{A}) = \|a\|_{A_0}, \qquad \lim_{t\to 0} \frac{1}{t} K(t, a; \bar{A}) = \|a\|_{A_1}.$$

In fact as we shall see next, the extra condition we need to effect control to prove the analogue of Theorem 5 in the general case is a uniform condition on the Gagliardo closure of the sequence (cf. Remark 8 below).

THEOREM 5.7. *Let* (A_0, A_1) *be a regular and mutually closed pair, let* $a \in A_0$, $\{a_n\}_{n\in N} \subset A_0$, *then*

$$a_n \xrightarrow{A_0} a \Leftrightarrow \begin{cases} \text{(i)} & \lim_{n\to\infty} -\tilde{E}'(s, a-a_n) = 0, \qquad s>0 \\ \text{(ii)} & \{a_n\}_{n\in N} \text{ is } K\text{-uniformly continuous} \\ \text{(iii)} & \lim_{t\to\infty} \sup_n \int_t^\infty K'(s, a_n - a; \bar{A})\, ds = 0. \end{cases}$$

Proof. The proof is almost identical to the proof of Theorem 5. To see the "*if part*" note that if $\bar{A}$ is regular and mutually closed then

$$\int_0^\infty K'(s, a; \bar{A})\, ds = \|a\|_{A_0} = \sup_{t>0} K(s, a; \bar{A})$$

so (i) and (ii) follow as in Theorem 5, while (iii) is proved in the same way as (ii). To see the converse, given $\varepsilon > 0$ by condition (iii) $\exists t_0 > 0$ such that $\forall n \in N$

$$\int_{t_0}^\infty K'(s, a - a_n; \bar{A})\, ds < \frac{\varepsilon}{3}.$$

By (ii) we can choose $t_1 \leqslant t_0$ so that $\forall n \in N$,

$$K(t_1, a_n - a) < \frac{\varepsilon}{3}.$$

Finally by (i) select n_0 such that $\forall n > n_0$,

$$-\tilde{E}'\left(\frac{\varepsilon}{3t_0}, a - a_n\right) < t_1.$$

Now, let us write

$$\begin{aligned}\|a-a_n\|_{A_0} &= \int_{\{s\in(0,t_0):\,K'(s,a-a_n)>\varepsilon/3t_0\}} K'(s,a-a_n)\,ds\\ &\quad+\int_{\{s\in(0,t_0):\,K'(s,a-a_n)\leqslant\varepsilon/3t_0\}} K'(s,a-a_n)\,ds\\ &\quad+\int_{t_0}^{\infty} K'(s,a-a_n)\,ds\\ &= I+II+III.\end{aligned}$$

Obviously

$$II\leqslant\varepsilon/3 \qquad \text{and} \qquad III\leqslant\varepsilon/3.$$

Finally I is controlled as in Theorem 5. ▮

Remark 7. If the pair $\bar{A}$ is regular and

$$\lim_{t\to\infty} K(t,a;\bar{A})=\|a\|_{A_0}$$

then the previous theorem remains true. If the pair is ordered then

$$K(t,a;\bar{A})=\|a\|_{A_0} \qquad \text{for all} \quad t\geqslant 1,$$

hence condition (iii) is obviously satisfied. In this case the condition that the pair be mutually closed can be dropped. If the pair (A_0, A_1) is such that the reversed pair (A_1, A_0) is ordered (which obviously implies that (A_0, A_1) is a regular pair) then $K(t,a;\bar{A})=t\,\|a\|_{A_1}$ for all $t\leqslant 1$ thus condition (ii) is equivalent to $\sup_n \|a_n\|_{A_1}<\infty$.

Remark 8. Note that for $a\in A_0$,

$$\int_t^{\infty} K'(s,a;\bar{A})\,ds=\|a\|_{A_0}-K(t,a;\bar{A}),$$

and by mutual closedness we always have

$$\|a\|_{A_0}-K(t,a;\bar{A})\to 0.$$

Condition (iii) is thus a uniform condition on the Gagliardo norm of $\{a_n-a\}_{n\in N}$. For suitable pairs this condition can be replaced by a condition on $\{a_n\}_{n\in N}$ only. This is the case, for example, if the pair $\bar{A}$ satisfies a condition of the form

$$K'(s,a_0+a_1;\bar{A})\leqslant c(K'(s/2,a_0;\bar{A})+K'(s/2,a_1;\bar{A})).$$

In particular, as is well known, this last condition holds for the pair (L^1, L^∞) (cf. also the proof of Theorem 9 below).

6. APPLICATIONS

6.1. *Lebesgue–Vitali Dominated Convergence Theorem.*

THEOREM 8. *Let* (Ω, μ) *be a finite measure space* $\{f_n\}_{n\in N} \subset L^1$, $f \in L^1$ *then*

$$f_n \xrightarrow{L^1} f$$

if and only if

$$\{f_n\}_n \qquad \text{is } K\text{-dominated and} \quad f_n \xrightarrow{m} f.$$

Proof. If (Ω, μ) has atoms we can embed (Ω, μ) into a non-atomic measure space $(\bar{\Omega}, \bar{\mu})$ (cf. [4], and [1] p. 54) such that for all μ-measurable function g on Ω

$$g_\mu^* = g_{\bar{\mu}}^*,$$

where the subscripts indicate the measure respect to which we take rearrangements. It follows that

$$f_n \xrightarrow{L^1(\Omega)} f \Leftrightarrow f_n \xrightarrow{L^1(\bar{\Omega})} f.$$

Therefore without loss of generality we may assume that (Ω, μ) is atom free and moreover $\mu(\Omega) = 1$. Consider now the ordered pair $(L^1(\Omega), L^\infty(\Omega))$, by Theorem 5 we know that

$$f_n \xrightarrow{L^1} f$$

if and only if

$$\{f_n\}_n \qquad \text{is } K\text{-uniformly continuous and} \quad \lim_{n\to\infty} -\tilde{E}'(t, f_n - f) = 0.$$

By Lemma 3-3, $\lim_{n\to\infty} -\tilde{E}'(t, f_n - f) = 0,\ t > 0$ iff $f_n \xrightarrow{m} f$. Therefore it remains to show that in the situation at hand K-uniform continuity is equivalent to K-domination. Using Ryff's theorem (cf. [1] pp. 82–86) we can further reduce ourselves to the case were $(\Omega, \mu) = ([0, 1], dx)$ in which case the argument we gave in the Introduction proves the result. An alternative proof can be based through an application of Theorem 2 and 3.

Indeed, again by Ryff's theorem, we can choose g to be a measurable function on (Ω, μ) such that (cf. [1] Corollary 7.8, p. 86)

$$g_\mu^*(t) = t^{-1/2}\chi[0,1](t).$$

Then,

$$K(t, g; L^1(\Omega), L^\infty(\Omega)) = \int_0^t g_\mu^*(s)\,ds = \int_0^t s^{-1/2}\chi_{[0,1]}(s)\,ds.$$

It follows that

$$K(t, g) \approx \int_0^t K(s, g)\frac{ds}{s} + t\int_t^\infty K(s, g)\frac{ds}{s^2}, \qquad 0 < t \leqslant 1.$$

Theorem 2 implies that $(L^1(\Omega), L^\infty(\Omega))$ is *Conv*$_{-}$*-abundant*, therefore we conclude the proof applying Theorem 3. ▮

For infinite measure spaces we have the following result

THEOREM 9. *Let* (Ω, μ) *be a measure space*, $\{f_n\}_{n\in N} \subset L^1$, $f \in L^1$ *then*

$$f_n \xrightarrow{L^1} f$$

if and only if

$$\lim_{t\to 0}\sup_{n\in N}\int_0^t f_n^* = 0, \qquad \lim_{t\to\infty}\sup_{n\in N}\int_t^\infty f_n^* = 0 \qquad \textit{and} \qquad f_n \xrightarrow{m} f.$$

Proof. Applying Theorem 7, and (1.2), it only remains to prove that

$$\lim_{t\to\infty}\sup_{n\in N}\int_t^\infty f_n^* = 0 \Leftrightarrow \lim_{t\to\infty}\sup_{n\in N}\int_t^\infty (f_n - f)^* = 0,$$

which follows readily using the well known inequality

$$(f+g)^*(s) \leqslant f^*(s/2) + g^*(s/2). \quad ▮$$

Remark 9. In the context of infinite measure spaces the classical condition at infinity that is imposed on $\{f_n\}_{n\in N}$ reads as follows: for all $\varepsilon > 0$ there exists a set E of finite measure such that $\sup_n \int_{E^c} |f_n(x)|\,dx < \varepsilon$. In comparing this condition with the one imposed in Theorem 9 note that for all $n \in N$, we trivially have $\int_{E^c} |f_n(x)|\,dx \geqslant \int_{|E|}^\infty f_n^*(s)\,ds$.

THEOREM 10. *Let* $\{f_n\}_n \subset \ell^1$, $f \in \ell^1$ *then*

$$f_n \xrightarrow{\ell^1} f$$

if and only if

$$\sup_n \|f_n\|_{\ell^1} < \infty, \qquad \lim_{m\to\infty} \sup_n \left(\sum_{j=m}^{\infty} (f_n)_j^* \right) = 0 \qquad \text{and} \qquad f_n \xrightarrow{m} f.$$

Proof. Consider the pair (ℓ^1, ℓ^∞), apply Theorem 7 and Remark 7. ∎

THEOREM 11. *Let* X *be a Banach lattice on a measure space* (Ω, μ) *such that the pair* (X, L^∞) *is an ordered regular pair. Let* $\{f_n\}_{n\in N} \subset X$, $f \in X$ *then*

$$f_n \xrightarrow{X} f$$

if and only if

$$\lim_{t\to\infty} \sup_{n\in N} \|[\,|f_n| - t]^+\|_X = 0, \qquad \text{and}$$

$$\lim_{n\to\infty} -\frac{\partial}{\partial t} \|[\,|f_n - f| - t]^+\|_X = 0, \qquad t > 0,$$

where $[f]^+ := \max(f, 0)$

Proof. It is well known that (cf. [3] Proposition 3.1.16)

$$\tilde{E}(t, f; X, L^\infty) = \|[\,|f| - t]^+\|_X.$$

Moreover, by Theorem 3, K-uniform continuity is equivalent to $\tilde{E}$-uniform continuity. The desired result now follows from Theorem 5. ∎

6.2. *Lebesgue–Vitali Convergence Theorem for noncommutative* L^p *spaces.* Let H be a Hilbert space, let S_∞ be the space of bounded operators from H to H. The Schatten ideals of operators S_p are defined as follows: A compact operator $T \in S_\infty$ is in the Schatten ideal S_p, $0 < p < \infty$, if

$$\|T\|_{S_p} = \|\{s_n(T)\}_n\|_{l^p} < \infty,$$

where $\{s_n(T)\}_n$ denotes the sequence of eigenvalues of the operator $(T^*T)^{1/2}$ arranged in decreasing order (i.e. $s_n(T) =$ *singular or s-numbers* of the operator T).

Define S_0 to be the space of operators $T \in S_\infty$ of finite rank $\|T\|_{S_0} = rank(T)$ (this is the analogue of the space L^0 of functions with finite support) then

$$s_n(T) = \inf\{\|T-R\|_{S_\infty} : \|R\|_{S_0} \leqslant n\} = \tilde{E}(n, T; S_0, S_\infty). \tag{6.1}$$

Note that the inverse of the function $s_n(T)$ is given by

$$v_n(T) = \inf\{\|T-R\|_{S_0} : \|R\|_{S_\infty} \leqslant n\} = E(n, T; S_0, S_\infty).$$

Moreover, it is well known that

$$K(t, T, S_1, S_\infty) = \int_0^t s_T(x)\, dx$$

where $s_T(x) := s_n(T)$ for $n \leqslant x < n+1$, $n \geqslant 1$. Thus

$$\lim_{t\to\infty} K(t, T, S_1, S_\infty) = \|T\|_{S_1}.$$

It follows from (6.1) that

$$s_{T_0+T_1}(x) \leqslant s_{T_0}(x/2) + s_{T_1}(x/2).$$

Now applying Theorem 7 to the pair (S_1, S_∞) we obtain (note that $S_1 \subset S_\infty$).

THEOREM 12. *Let* $\{T_m\}_{m\in N} \subset S_1$, $T \in S_1$, *then*

$$T_m \xrightarrow{S_1} T$$

if and only if

$$\sup_{m\in N} \|T_m\|_{S_1} < \infty,$$

$$\lim_{k\to\infty} \sup_{m\in N} \left\{\sum_{n=k}^{\infty} s_n(T_m)\right\} = 0 \qquad \textit{and} \qquad \lim_{m\to\infty} v_n(T_m - T) = 0, \qquad n \geqslant 1.$$

We can also apply our method in a slightly more general setting.

Our basic reference in what follows is [19].

Let H a Hilbert space, $\mathscr{A}$ a ring of operators on H. A *gage* on $\mathscr{A}$ is a mapping m: {projections of $\mathscr{A}$} $\to R^+$ such that

(1) $m(P) > 0$ if $P \neq 0$, $m(0) = 0$

(2) $m(\bigcup_\alpha P_\alpha) = \sum_\alpha P_\alpha$ if $P_\alpha P_\beta = 0$, $\alpha \neq \beta$

(3) $m(UPU^{-1}) = m(P)$ if $U^{-1} = U^*$

(4) every projection in $\mathscr{A}$ is $\bigcup$ of m-finite projections.

The triple $\Gamma=(H, \mathscr{A}, m)$ is called a *gage space*. Given a *gage space* Γ, we define the $L^p=L^p(\Gamma)$, $1\leqslant p\leqslant\infty$, (non-commutative L^p spaces) by the condition

$$\|T\|_{L^p}<\infty,$$

where, if $(T^*T)^{1/2}$ has spectral representation $\int_0^\infty \lambda\, dP(\lambda)$, then

$$\|T\|_{L^p}=\left(\int_0^\infty \lambda^p\, dm(P(\lambda))\right)^{1/p}.$$

We can also define L^0 with the norm $\|T\|_{L^0}=m(\operatorname{supp} T)$ where supp T is the smallest projection $P\in\mathscr{A}$ such that $PT=T$. Let

$$T^\star(t)=E(t, T; L^0, L^\infty)=\inf\{\|T-S\|_{L^\infty} : \|S\|_{L^0}\leqslant t\}.$$

Note that

$$(T_1+T_2)^\star\,(t)\leqslant T_1^\star(t/2)+T_2^\star(t/2),$$

Furthermore $t\to T^\star(t)$ is the inverse of the function $\lambda\to m(P(\lambda))$, where $P(\lambda)$ is the spectral resolution of $(T^*T)^{1/2}$, and we have

$$K(t, T; L^1, L^\infty)=\int_0^t T^\star(s)\, ds.$$

Obviously

$$\lim_{t\to 0} K(t, T; L^1, L^\infty)=0 \quad\text{and}\quad \lim_{t\to\infty} K(t, T; L^1, L^\infty)=\int_0^\infty T^\star(s)\, ds=\|T\|_{L^1}.$$

A direct application of Theorem 7 yields

THEOREM 13. *Let $\{T_n\}_{n\in N}\subset L^1(\Gamma)$, $T\in L^1(\Gamma)$ then*

$$T_n \xrightarrow{L^1(\Gamma)} T$$

if and only if

$$\lim_{t\to 0}\sup_{n\in N}\int_0^t T_n^\star(s)\, ds=0,$$

$$\lim_{t\to\infty}\sup_{n\in N}\int_t^\infty T_n^\star(s)\, ds=0 \qquad \textit{and} \qquad \lim_{n\to\infty} m(P_n(\lambda))=0, \qquad \lambda>0,$$

where $P_n(\lambda)$ is the spectral resolution of $((T-T_n)^*\circ(T-T_n))^{1/2}$.

6.3. *A version of Lebesgue–Vitali in the context of certain variational problems.* As a new application of the methods developed in this paper we now prove convergence theorems in the context of the theory of variational problems studied by Micchelli–Pinkus [14]. We start by reviewing the basic definitions of the Michelli–Pinkus theory.

Let X be a normed space, let T be a compact Haussdorf space and let $K \subset X$, be a convex subset. We consider a family of real valued functions $G_t(x) = G(t, x)$, $x \in X$, $t \in T$ satisfying

1. $$\sup_{t \in T} |G(t, x)| < \infty, \qquad x \in X.$$

2. For each $t \in T$,

$$x \to G_t(x) \qquad \text{is convex on } K.$$

3. For any given $x, y \in K$ such that $G(t, x) < G(t, y)$ $\forall t \in T$, $\exists c > 0$ such that

$$0 < c \leqslant G(t, y) - G(t, x), \qquad t \in T.$$

Let us say that $x_0 \in K$ is a best G-approximation from K if $\nexists x \in K$ such that

$$G(t, x) < G(t, x_0), \qquad \forall t \in T.$$

Then it is shown in [14] that $x_0 \in K$ is a best G-approximation if and only if there exists a nonnegative nontrivial linear functional L on the space $B(T)$ of real valued bounded functions defined on T, such that

$$L(G(., x_0)) = \min_{x \in K} L(G(., x)).$$

This result reduces G-approximation to minimization of a convex function (namely $L(G(., x))$).

If $T = \{0, 1\}$ then we only have two functionals G_1 and G_2, say. In this case to find the best G-approximation is equivalent to the minimization problem

$$\inf\{\sigma_1 G_1(x) + \sigma_2 G_2(x), \sigma_i \geqslant 0, i = 1, 2, \sigma_1 + \sigma_2 = 1\}.$$

This leads directly to the definition

$$(G_1 + G_2)(\sigma) := \inf_{x \in K} \{G_1(x) + \sigma G_2(x)\}, \qquad \sigma > 0, \tag{6.2}$$

as well as the functionals

$$(G_1/G_2)(\sigma) := \inf_{x \in K} \{G_1(x) : G_2(x) \leqslant \sigma\}, \tag{6.3}$$

where $\sigma > \mu(G_2) = \inf_{x\in K} G_2(x)$, and

$$(G_2/G_1)(\sigma) := \inf_{x\in K} \{G_2(x) : G_1(x) \leqslant \sigma\}, \tag{6.4}$$

where $\sigma > \mu(G_1) = \inf_{x\in K} G_1(x)$.

We can also define these functionals at the left endpoints of their domain of definition by considering right-limits, i.e.

$$\mu[G_1 + G_2] = \lim_{\sigma\to 0^+} (G_1 + G_2)(\sigma);$$

$$\mu[G_1; G_2] = \lim_{\sigma\to \mu(G_2)^+} (G_1/G_2)(\sigma)$$

(similarly we define $\mu[G_2; G_1]$).

When thinking about the correspondence with interpolation theory we should keep in mind that if we fix $a \in X$, and if $\|\cdot\|_2$ is another norm defined on X, then letting

$$G_1(x) = \|a - x\|_X, \qquad G_2(x) = \|x\|_2$$

we shall be in the usual setting of interpolation theory in which case we recover the K and E-functionals.

It will be convenient also to assume that G_1, G_2 are bounded below, so that without loss one may assume that both functionals are nonnegative.

Now to these functionals we associate a Gagliardo diagram

$$\Gamma = \{(y_1, y_2) \in \mathbf{R}^2 : G_i(x) \leqslant y_i,\ i = 1, 2, \text{ for some } x \in K\}. \tag{6.5}$$

In the next Lemma we collect results from [14] showing that the behavior of these functionals is almost identical to the behavior of the $E-K$-functionals of interpolation theory.

Lemma 6 ([14] Theorem 2.2, Proposition 2.3). (1) *$(G_1/G_2)(\sigma)$ is decreasing convex on its domain of definition and continuous on the interior.*

(2) *$(G_1/G_2)(\sigma) = \mu(G_1)$ if $\sigma > \mu[G_2; G_1]$.*

(3) *$(G_2/G_1)((G_1/G_2)(\sigma)) = \sigma$ for $\sigma \in (\mu(G_2), \mu[G_2; G_1])$.*

(4) *$(G_1 + G_2)(\sigma) = \inf_{t>\mu(G_1)}(t + \sigma(G_2/G_1)(t)) = \inf_{t>\mu(G_2)}(t\sigma + (G_1/G_2)(t))$.*

(5) *For $t > \mu(G_1)$*

$$(G_2/G_1)(t) = \sup_{\sigma>0} \left(\frac{(G_1 + G_2)(\sigma) - t}{\sigma}\right).$$

(6) *If* $\mu(G_2)=0$

$$(G_1/G_2)(t)=\sup_{\sigma>0}((G_1+G_2)(\sigma)-\sigma t).$$

(7) $(G_1+G_2)(\sigma)$ *is a increasing continuous concave function,* $\frac{(G_1+G_2)(\sigma)}{\sigma}$ *is decreasing. Furthermore if there exists* $x^*\in K$ *such that* $G_2(x^*)=0$, *then* $(G_1+G_2)(\sigma)$ *is bounded.*

(8) $\mu[G_1+G_2]=\lim_{\sigma\to 0^+}(G_1+G_2)(\sigma)=\mu(G_1)$ *and* $\lim_{\sigma\to\infty}\frac{(G_1+G_2)(\sigma)}{\sigma}$ $=\mu(G_2)$.

Proof. Except for 6 all other statements are contained in ([14], Theorem 2.2, and Proposition 2.3). To see 6 using the second equality in 4, (3.12) and the fact that (G_1/G_2) is convex, we see that

$$\begin{aligned}\sup_{\sigma>0}(\inf_{s>0}(s\sigma+(G_1/G_2)(s))-\sigma t)&=((G_1/G_2)^{\nabla})^{\Delta}(t)\\ &=(G_1/G_2)(t).\quad\blacksquare\end{aligned}$$

In what follows we assume that

(1) $(G_1/G_2)(\sigma)$ and $(G_2/G_1)(\sigma)$ are finite.

(2) $\exists x^*\in K$ such that $G_2(x^*)=0$, (this condition implies that G_1+G_2 and G_1/G_2 are well defined on $[0,\infty)$).

In this context we have a perfect analogue of (3.5)–(3.8).

LEMMA 7. *The following relations hold*

$$\sigma=(G_1/G_2)'(s);\qquad (G_1+G_2)(\sigma)=(G_1/G_2)(\sigma)-\sigma(G_1/G_2)'(s).\tag{6.6}$$

$$s=(G_1+G_2)'(\sigma);\qquad (G_1/G_2)(s)=(G_1+G_2)(\sigma)-s(G_1+G_2)'(\sigma).\tag{6.7}$$

We can give a meaning to (6.6) *and* (6.7) *even when the derivative does not exist using a suitable values between the left and right derivative of* (G_1/G_2) *and* (G_1+G_2). *In particular* $(G_1+G_2)'$ *and* $-(G_1/G_2)'$ *are inverse to each other.*

Proof. Note that (6.6) and (6.7) will follow from (3.5) and (3.8) (cf. [11, 13]) if we can prove that there exists a pair $\bar{A}=(A_0,A_1)$ such that for some $g\in A_0+A_1$ and $\forall\sigma>0$

$$(G_1+G_2)(\sigma)=K(\sigma,g;\bar{A})\qquad\text{and}\qquad (G_1/G_2)(\sigma)=\tilde{E}(\sigma,g;\bar{A}).$$

This can be seen as follows. Since (G_1+G_2) is concave, and the K-functional for the pair $(L^\infty, L^\infty(\frac{1}{t}))$ reproduces concave functions (cf. [3], Proposition 3.1.17), we have

$$(G_1+G_2)(\sigma)=K\left(\sigma, (G_1+G_2); L^\infty, L^\infty\left(\frac{1}{t}\right)\right).$$

Moreover,

$$\begin{aligned}\tilde{E}\left(\sigma, (G_1+G_2); L^\infty, L^\infty\left(\frac{1}{t}\right)\right) &= \sup_{s>0}((G_1+G_2)(s)-\sigma s)\\ &=(G_1/G_2)(\sigma) \qquad \text{(by Lemma 6–6).}\end{aligned}$$ ▮

If we combine the previous Lemma with Theorem 5 we get

THEOREM 6.10. *Let* $\{G_1^n\}_{n\in N}$, G_2 *be convex nonnegative functions on a subset* K *of a given linear space* X. *Suppose that* $\exists x^* \in K$ *such that* $G_2(x^*)=0$, *and for all* $n\in N$ $\inf_{x\in K}\{G_1^n(x) : G_2(x)\leqslant\sigma\}$ *is well defined. Then if* $0<\sigma_0<\infty$,

(1)
$$\left.\begin{aligned}&\lim_{\sigma\to 0}\sup_{n\in N}[(G_1^n+G_2)(\sigma)-\mu(G_1^n)]=0\\ &\lim_{n\to\infty}-(G_1^n/G_2)'(\sigma)=0, \qquad \forall\sigma>0\end{aligned}\right\}$$
$$\Rightarrow \lim_{n\to\infty}[(G_1^n+G_2)(\sigma_0)-\mu(G_1^n)]=0$$

(2)
$$\lim_{n\to\infty}[(G_1^n)(x^*)-\mu(G_1^n)]=0$$
$$\Rightarrow \begin{cases}\lim_{\sigma\to 0}\sup_{n\in N}[(G_1^n+G_2)(\sigma)-\mu(G_1^n)]=0\\ \lim_{n\to\infty}-(G_1^n/G_2)'(\sigma)=0, \qquad \forall\sigma>0\end{cases}$$

(3) *If there exists* $\sigma^*>0$ *such that* $(G_1^n+G_2)(\sigma)=(G_1^n+G_2)(\sigma^*)$ $\forall\sigma\geqslant\sigma^*$ *then if* $0<\sigma_0\leqslant\sigma^*$,

$$\lim_{n\to\infty}[(G_1^n+G_2)(\sigma_0)-\mu(G_1^n)]=0$$

$$\Leftrightarrow \begin{cases}\lim_{\sigma\to 0}\sup_{n\in N}[(G_1^n+G_2)(\sigma)-\mu(G_1^n)]=0\\ \lim_{n\to\infty}-(G_1^n/G_2)'(\sigma)=0, \qquad \forall\sigma\in(0,\sigma^*)\end{cases}.$$

Before outlining the proof we discuss a few examples.

(1) *Let* $\bar{X}=(X_0, X_1)$ *be a pair, and let* $\{a_n\}_{n\in N}\subset X_0$, $a\in X_0$, $K=X_0+X_1$, $G_1^n(x)=\|(a_n-a)-x\|_{X_0}$ *and* $G_2(x)=\|x\|_{X_1}$ *then* $(G_1^n+G_2)(\sigma)=K(\sigma, a_n-a; \bar{X})$ *and* $(G_1^n/G_2)(\sigma)=\tilde{E}(\sigma, a_n-a; \bar{X})$. *Moreover, in this case* $x^*=0$, *since*

$$G_2(0)=\|0\|_{X_1}=0,$$

then by Lemma 6–8,

$$\mu(G_1^n)=\lim_{\sigma\to 0}(G_1^n+G_2)(\sigma)=\lim_{\sigma\to 0}K(\sigma, a_n-a; \bar{X}),$$

where

$$\mu(G_1^n)=\inf_{x\in X_1}\|(a_n-a)-x\|_{X_0}=d(a_n-a, \overline{X_0\cap X_1}^{X_0})$$

(*here* $d\equiv$ *distance*), *and*

$$(G_1^n+G_2)(\sigma)\leqslant\|a_n-a\|_{X_0}=(G_1^n)(0).$$

Then the right hand side of 1 *is equivalent to*

$$\lim_{n\to\infty}[K(\sigma_0, a_n-a)-d(a_n-a, \overline{X_0\cap X_1}^{X_0})]=0,$$

while the left hand side of 2 *is equivalent to*

$$\lim_{n\to\infty}[\|a_n-a\|_{X_0}-d(a_n-a, \overline{X_0\cap X_1}^{X_0})]=0.$$

(2) *If the pair* (X_0, X_1) *is ordered* (*in which case can take* $K=X_1$) *then*

$$(G_1^n+G_2)(\sigma)=\|a_n-a\|_{X_0}=(G_1^n)(0), \qquad \forall\sigma\geqslant 1$$

and now 1 *and* 2 *are equivalent.*

(3) *If the pair* (X_0, X_1) *is ordered and regular then* $d(a_n-a, \overline{X_0\cap X_1}^{X_0})=0$, *and in this case the result includes Theorem* 5.

(4) *In the setting of Micchelli–Pinkus* (*cf.* [14] *Chapt.* 3) *we can also consider* $G_1^n(g)=\|f-T_n^*g\|_{X^*}$ *and* $G_2(g)=\|g\|_{Y^*}$.

Proof. The proof is a small modification of the proof of Theorem 5. We indicate briefly the changes needed leaving the details to the interested reader.

1. For a fixed $0<\sigma_0<\infty$ we write

$$(G_1^n+G_2)(\sigma_0)-\mu(G_1^n)=\int_0^{\sigma_0}(G_1^n+G_2)'(\xi)\,d\xi.$$

Now we finish as in Theorem 5.

To see (2) we apply Lemma 7 to get

$$\begin{aligned}\sup_\sigma\,[-\sigma(G_1^n/G_2)'(\sigma)]&=\sup_\sigma\,[\sigma(G_1^n+G_2)'(\sigma)]\\&\leqslant\sup_\sigma\int_0^\sigma(G_1^n+G_2)'(\xi)\,d\xi\\&\qquad\text{(since }(G_1^n+G_2)'\text{ decreases)}\\&\leqslant\sup_\sigma\,[(G_1^n+G_2)(\sigma)-\mu(G_1^n)]\\&\leqslant(G_1^n)(x^*)-\mu(G_1^n)\\&\qquad\text{(since }(G_1^n+G_2)(\sigma)\leqslant(G_1^n)(x^*)).\end{aligned}$$

We may now continue as in Theorem 5. ∎

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REFERENCES

1. C. Bennett and R. Sharpley, "Interpolation of Operators," Academic Press, Boston, 1988.
2. J. Bergh and J. Lofström, "Interpolation Spaces, An Introduction," Springer-Verlag, New York, 1976.
3. Yu. A. Brudnyi and N. Ya. Krugljak, "Interpolation Functors and Interpolation Spaces," North-Holland, Amsterdam, 1991.
4. A. P. Calderón, Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz, *Studia Math.* **26** (1966), 273–299.
5. J. Chaumat, Quelques proprietes du couple d'espaces vectoriels $(L^1(m)/H^{\infty\perp}, H^\infty)$, preprint, University of Paris-Sud, Orsay, 1978.
6. L. E. Clarke, Problem Solution 6174, *Amer. Math. Monthly* **86** (1979), 313.
7. N. Dunford and B. J. Pettis, Linear operators on summable functions, *Trans. Amer. Math. Soc.* **47** (1940), 323–392.
8. G. A. Edgar and L. Sucheston, "Stopping Times and Directed Processes," Cambridge Univ. Press, Cambridge, UK, 1992.
9. G. B. Folland, "Real Analysis," Wiley, New York, 1984.
10. A. Grothendieck, "Topological Vector Spaces," Gordon and Breach, New York, 1973.

11. T. Holmstedt, Equivalence of two methods of interpolation, *Math Scand.* **18** (1966), 45–52.
12. B. Jawerth, R. Rochberg, and G. Weiss, Commutator and other second order estimates in real interpolation theory, *Ark. Mat.* **24** (1986), 191–219.
13. B. Jawerth and M. Milman, Interpolation of weak type spaces, *Math. Z.* **201** (1989), 509–519.
14. C. A. Micchelli and A. Pinkus, Variational problems arising from balancing several error criteria, *Rend. Mat.* **14** (1994), 37–86.
15. M. Milman, "Extrapolation and Optimal Decompositions with Applications to Analysis," Lecture Notes in Math., Vol. 1580, Springer-Verlag, New York, 1994.
16. M. Milman and R. Rochberg, The role of cancellation in interpolation theory, Harmonic analysis and operator theory, *Contemp. Math.* **189** (1995), 403–419.
17. E. T. Oklander, "On Interpolation of Banach Spaces," Ph.D. thesis (under A. P. Calderón), University of Chicago, 1963. [See also E. T. Oklander, Interpolación, espacios de Lorentz y teorema de Marcinkiewicz, Cursos y Seminarios de Matematica 20, University of Buenos Aires, 1965.]
18. J. Peetre, A theory of Interpolation of Normed Spaces, *Notas de Matematica*, *Rio de Janeiro* **39** (1963).
19. J. Peetre and G. Sparr, Interpolation and non commutative integration, *Ann. Mat. Pura Appl.* **104** (1975), 187–207.
20. M. M. Rao and Z. D. Ren, Theory of Orlicz Spaces, *Pure Appl. Math.* **146** (1991).